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INTEGRAL METHODS IN TRANSIENT HEAT CONDUCTION PROBLEMS WITH NON-UNIFORM INITIAL CONDITIONS

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INTRODUCTION

Various authors [1–6] have applied the heat balance integral method to one-dimensional conduction problems, but in every case the applications have been limited to uniform initial temperature conditions. The only reference to non-uniform conditions is contained in a suggestion by Yang [7], who proposed the introduction of a new temperature excess variable θ^+.

If, for instance, in the original differential equation applicable to a slab $0 \leqslant x \leqslant L$

$$\frac{\partial \theta}{\partial t} = \frac{a}{L^2}\frac{\partial^2 \theta}{\partial \eta^2} \tag{1}$$

with initial conditions $\theta(\eta, 0) = f(\eta)$ and suitable boundary conditions, the excess temperature $\theta^+ = \theta(\eta, t) - \theta(\eta, 0)$ is introduced, the differential equation becomes

$$\frac{\partial \theta^+}{\partial t} = \frac{a}{L^2}\left(\frac{\partial^2 \theta^+}{\partial \eta^2} + \frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2}\right) \tag{2}$$

Here θ is temperature, a the thermal diffusivity, t is time, L the slab thickness and $\eta = x/L$. $f(\eta)$ is assumed to be a specified function. Although Yang's suggestion works in a few simple cases, it suffers from three disadvantages. First, the addition of $(a/L^2)(\mathrm{d}^2f/\mathrm{d}\eta^2)$ generally complicates the basic integral procedure. Second, the assumed temperature profile representation is limited to at most a fourth degree polynomial in η, since only five independent conditions are available for determining the time dependent polynomial coefficients. These are the two boundary conditions, the two relations derived by substituting the assumed profile into (2) and evaluating the resulting expression at the boundaries, and the heat balance integral. Finally, the most serious limitation is that generally some of the resulting five relations are differential equations, which will not permit the direct evaluation of the constants of integration so that auxiliary techniques will have to be introduced if the profile is to be of higher than second degree.

The purpose of the present note is to suggest two alternative techniques which not only avoid these difficulties but also allow a direct method of attack.

DEVELOPMENT OF "MOMENT" AND "SUBDIVISION" METHODS

Assuming the solution of (1) can be expressed by a polynomial of the form

$$\theta(\eta, t) = \sum_{k=0}^{n} a_k(t)\eta^k \tag{3}$$

$n + 1$ relations are needed to determine the coefficients $a_k(t)$. Two relations are obtained by expressing the boundary conditions at $\eta = 0$ and $\eta = 1$ in terms of (3). The $n - 1$ additional equations required can be obtained either by the so-called "moment method" or by a "subdivision method".

The moment method, which is a standard mathematical technique, was discussed by Costello [8] for conduction problems with uniform initial conditions and for which the concept of a penetration thickness remained applicable.

Multiplying (1) by a weighting function $g_i(\eta)$ and integrating the resulting equation between $\eta = 0$ and $\eta = 1$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 g_i(\eta)\theta(\eta, t)\,\mathrm{d}\eta = \frac{a}{L^2}\int_0^1 g_i(\eta)\frac{\partial^2\theta}{\partial\eta^2}\,\mathrm{d}\eta \tag{4}$$

Substituting (3) into (4) results in

$$\sum_{k=0}^{n}\left[\int_0^1 g_i(\eta)\eta^k\,\mathrm{d}\eta\right]\cdot\frac{\mathrm{d}a_k}{\mathrm{d}t} = \frac{a}{L^2}\sum_{k=2}^{n}\left[\int_0^1 k(k-1)g_i(\eta)\eta^{k-2}\,\mathrm{d}\eta\right]\cdot a_k \tag{5}$$

A system of $n - 1$ linearly independent first order ordinary differential equations is obtained by choosing $n - 1$ different $g_i(\eta)$, $(i = 1, 2, \ldots, n - 1)$. The constants of integration resulting from the solution to this system are determined by equating $n - 1$ moments of $f(\eta)$ to the corresponding moments of $\theta(\eta, 0)$ of equation (3), i.e.

$$\int_0^1 g_i(\eta)f(\eta)\,\mathrm{d}\eta = \int_0^1 g_i(\eta)\sum_{k=0}^{n} a_k(0)\eta^k\,\mathrm{d}\eta \tag{6}$$

The solution will then become unique once the set $\{g_i(\eta)\}$ is specified. Since the temperature distribution is assumed to be a polynomial, the selection of $g_i(\eta)$ as an $(i-1)$th degree polynomial appears to be natural. It may be easily verified that the solution will indeed be unique irrespective of the form of the polynomial $g_i(\eta)$ as long as it is of degree $(i-1)$. Thus the set $\{g_i(\eta)\}$ may be selected as $\{\eta^{i-1}\}$, $(i = 1, 2, \ldots, n-1)$, for convenience.

Another systematic method which can be used for determining the a_k's is the "subdivision method", which was also suggested in reference 8. The slab is divided into $n-1$ equal subdivisions. Integrating (1) between $\eta = \lambda$ and $\eta = \mu$ yields

$$\frac{d}{dt}\int_{\lambda}^{\mu} \theta \, d\eta = \frac{\alpha}{L^2}\left[\frac{\partial\theta}{\partial\eta}\bigg|_{\mu} - \frac{\partial\theta}{\partial\eta}\bigg|_{\lambda}\right]. \tag{7}$$

Substituting (3) into (7) leads to

$$\sum_{k=0}^{n}\left[\frac{\mu^{k+1}-\lambda^{k+1}}{k+1}\right]\frac{da_k}{dt} = \frac{\alpha}{L^2}\sum_{k=1}^{n} k\left[\mu^{k-1}-\lambda^{k-1}\right]a_k \tag{8}$$

A system of $n-1$ linearly independent first order ordinary differential equations is obtained by selecting λ and μ of equation (7) so that integration is performed over each of the $n-1$ subdivisions. The solution to this system results in $n-1$ constants of integration which can be evaluated by equating the integral of $f(\eta)$ and the integral of $\theta(\eta, 0)$ from (3) over each of the subdivisions. There appears to be little *a priori* reason for the choice of either of the two methods discussed over the other; the results of both will approach the exact solution as the degree of the polynomial is increased.

ILLUSTRATIVE EXAMPLES

Two illustrative examples indicating typical applications are discussed below. The first represents a finite slab with an initial linear temperature distribution which is suddenly insulated at the ends at $t > 0$ and is shown on Fig. 1. This example is representative of a class of problems which exhibit a discontinuity at $t = 0$. Since two of the coefficients a_k are determined by satisfying the boundary conditions at all times, difficulty in matching initial conditions is to be expected. It then becomes of interest to determine how rapidly the integral solution approaches the exact solution, which must be expressed by an infinite number of terms. However, for a desired degree of accuracy the number of terms required decreases rapidly with time and an early agreement between the exact and approximate solutions might be expected.

Applying the initial conditions and boundary conditions

$$\theta(\eta, 0) = f(\eta) = \theta_2 \cdot \eta \tag{9a}$$

$$\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=0} = \left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1} = 0 \tag{9b}$$

and using a cubic profile leads to a solution by the "moment method" of

$$\frac{\theta(\eta, t)}{\theta_2} = \frac{1}{2} - 5\exp\left[-10(\alpha/L^2)t\right]\left[\frac{1}{12} - \frac{1}{2}\eta^2 + \frac{1}{3}\eta^3\right]. \tag{10}$$

Application of the "subdivision method" yields

$$\frac{\theta(\eta, t)}{\theta_2} = \frac{1}{2} - \exp\left[-9{\cdot}6(\alpha/L^2)t\right]\left[\frac{2}{5} - \frac{12}{5}\eta^2 + \frac{8}{5}\eta^3\right]. \tag{11}$$

Figure 1 indicates that equations (10) and (11) compare favorably with the exact solution given in reference 9 for $(\alpha/L^2)t \geqslant 0{\cdot}05$. For $(\alpha/L^2)t$ closer to zero there will be greater error because of failure to satisfy the initial condition exactly.

The second example selected is characteristic of problems where the initial temperature distribution becomes highly distorted at even very short values of time. Here the slab with the initial parabolic temperature distribution

$$\theta(\eta, 0) = f(\eta) = -4\theta_{\max}(\eta^2 - \eta) \tag{12a}$$

is subject to the boundary conditions

$$\theta(0, t) = \theta_0 = \theta_{\max}(1 - \exp\left[-100(\alpha/L^2)t\right]) \tag{12b}$$

$$\theta(1, t) = 0 \tag{12c}$$

Since the selection of a step input boundary condition would have led to both rapid profile distortion and to a discontinuity at $t = 0$, the effect of the discontinuity was suppressed by the selection of a continuous but rapidly varying boundary condition.

Again assuming a cubic profile the "moment" solution is

$$\left.\begin{aligned}\frac{\theta(\eta, t)}{\theta_{\max}} = {} & (1 - \exp\left[-100(\alpha/L^2)t\right])(1-\eta) - 0{\cdot}591 \\ & \exp\left[-12(\alpha/L^2)t\right](\eta^2 - \eta) - \\ & 25\exp\left[-60(\alpha/L^2)t\right]\left(\frac{1}{2}\eta - \frac{3}{2}\eta^2 + \eta^3\right) + \\ & \exp\left[-100(\alpha/L^2)t\right](15{\cdot}909\eta - \\ & 40{\cdot}909\eta^2 + 25\eta^3)\end{aligned}\right\} \tag{13}$$

and the "subdivision" solution is

$$\left.\begin{aligned}\frac{\theta(\eta, t)}{\theta_{\max}} = {} & (1 - \exp\left[-100(\alpha/L^2)t\right])(1-\eta) - \\ & 0{\cdot}590\exp\left[-12(\alpha/L^2)t\right](\eta^2 - \eta) - \\ & 15{\cdot}392\exp\left[-48(\alpha/L^2)t\right]\left(\frac{1}{2} - \frac{3}{2}\eta^2 + \eta^3\right) + \\ & \exp\left[-100(\alpha/L^2)t\right](11{\cdot}101\eta - 26{\cdot}486\eta^2 + \\ & 15{\cdot}385\eta^3).\end{aligned}\right\} \tag{14}$$

The exact solution obtained from [9] is

$$\frac{\theta(\eta, t)}{\theta_{\max}} = (1 - \exp[-100(a/L^2)t])(1 - \eta)$$

$$+ \sum_{n=1}^{\infty} \left\{\frac{16[1 - (-1)^n]}{n^3\pi^3} - \frac{2}{n\pi}\right\} \exp[-n^2\pi^2(a/L^2)t] \sin(n\pi\eta)$$

$$+ 2\sum_{n=1}^{\infty} \left[\frac{n\pi \exp[-n^2\pi^2(a/L^2)t] - (100/n\pi)\exp[-100(a/L^2)t]}{n^2\pi^2 - 100}\right] \sin(n\pi\eta). \tag{15}$$

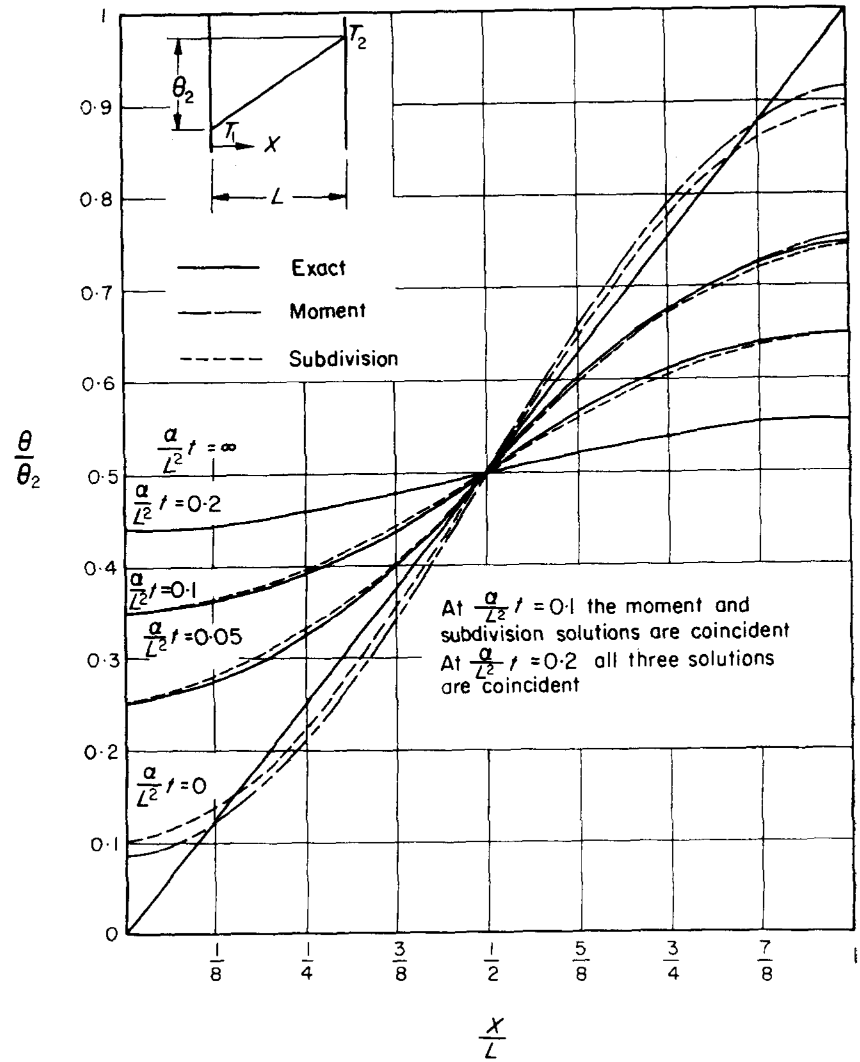

FIG. 1. Comparison of exact and integral solutions to example 1.

Graphs of equations (13), (14) and (15) are compared in Fig. 2. Though the integral solutions agree qualitatively with the exact solution there is considerable error even at $(a/L^2)t = 0{\cdot}10$. Considerable improvement can be obtained by increasing the degree of the polynomial to four. The "moment" solution then becomes, (also shown on Fig. 2)

The good agreement with the exact solution can also be noted from an examination of the dominant exponential coefficient which is $-9{\cdot}8751$ in the quartic and $-9{\cdot}8696$ in the exact solution. These examples may be taken as an indication of the utility and possible accuracy of the proposed methods.

$$\begin{aligned}\frac{\theta(\eta, t)}{\theta_{\max}} = & (1 - \exp[-100(a/L^2)t])(1 - \eta) + \\ & \exp[-9{\cdot}8751(a/L^2)t][1{\cdot}1014\eta^4 - 2{\cdot}2028\eta^3 + 0{\cdot}07769\eta^2 + 1{\cdot}0237\eta] + \\ & \exp[-60(a/L^2)t][-25{\cdot}0005\eta^3 + 37{\cdot}5007\eta^2 - 12{\cdot}5002\eta] + \\ & \exp[-170{\cdot}1249(a/L^2)t][-56{\cdot}4819\eta^4 + 112{\cdot}964\eta^3 - 68{\cdot}6355\eta^2 + 12{\cdot}1534\eta] + \\ & \exp[-100(a/L^2)t][55{\cdot}3797\eta^4 - 85{\cdot}7595\eta^3 + 27{\cdot}0570\eta^2 + 3{\cdot}2278\eta] \end{aligned} \quad (16)$$

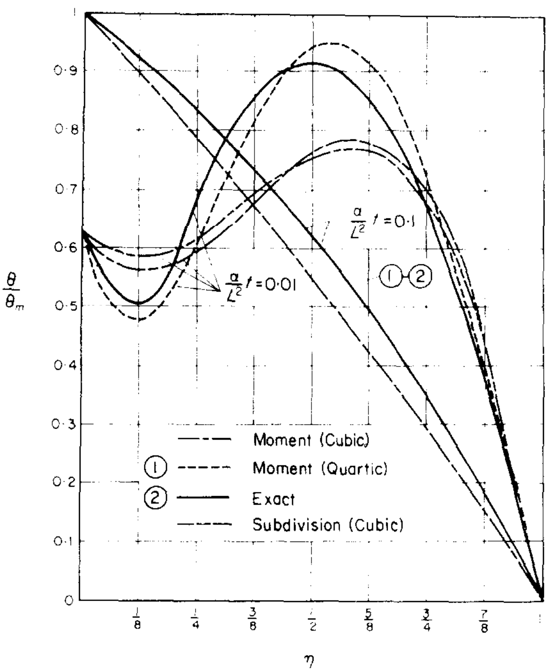

FIG. 2. Comparison of exact and integral solutions to example 2.

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